A supergeometric interpretation of vertex operator superalgebras

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1 Introduction

Conformal field theory (or more specifically, string theory) and related theories (cf. [BPZ], [FS], [V], and [S]) are the most promising attempts at developing a physical theory that combines all fundamental interactions of particles, including gravity. The geometry of this theory extends the use of Feynman diagrams, describing the interactions of point particles whose propagation in time sweeps out a line in space-time, to one-dimensional "particles" (strings) whose propagation in time sweeps out a two-dimensional surface. For genus zero holomorphic conformal field theory, algebraically, these interactions can be described by products of vertex operators or more precisely, by means of vertex operator algebras (cf. [Bo] and [FLM]). However, until 1990 a rigorous mathematical interpretation of the geometry and algebra involved in the "sewing" together of different particle interactions, incorporating the analysis of general analytic coordinates, had not been realized. In [H1] and [H2], motivated by the geometric notions arising in conformal field theory, Huang gives a precise geometric interpretation of the notion of vertex operator algebra by considering the geometric structure consisting of the moduli space of genus zero Riemann surfaces with punctures and local coordinates vanishing at the punctures, modulo conformal equivalence, together with the operation of sewing two such surfaces, defined by cutting discs around one puncture from each sphere and appropriately identifying the boundaries. Important aspects of this geometric structure are the concrete realization of the moduli space in terms of exponentials of a representation of the Virasoro algebra and a precise analysis of sewing using these resulting exponentials. Using this geometric structure, Huang then introduces the notion of geometric vertex operator algebra with central charge $c \in \mathbb{C}$, and proves that the category of geometric vertex operator algebras is isomorphic to the category of vertex operator algebras.

In [F], Friedan describes the extension of the physical model of conformal field theory to that of superconformal field theory and the notion of a superstring whose propagation in time sweeps out a supersurface. Whereas conformal field theory attempts to describe the interactions of bosons, superconformal field theory attempts to describe the interactions of boson-fermion pairs. This, in particular, requires an operator D such that $D^2 = \frac{\partial}{\partial z}$. Such an operator arises naturally in supergeometry. In [BMS], Beilinson, Manin and Schechtman study some aspects of superconformal symmetry, i.e., the Neveu-Schwarz algebra, from the viewpoint of algebraic geometry. In this work, we will take a differential geometric approach, extending Huang's geometric interpretation of vertex operator algebras to a supergeometric interpretation of vertex operator superalgebras.

Within the framework of supergeometry (cf. [D], [R] and [CR]) and motivated by superconformal field theory, we define the moduli space of super-Riemann surfaces with genus zero "body", punctures, and local superconformal coordinates vanishing at the punctures, modulo superconformal equivalence. We announce the result that any local superconformal coordinates can be expressed in terms of exponentials of certain superderivations, and that these superderivations give a representation of the Neveu-Schwarz algebra with zero central charge. We define a

sewing operation on this moduli space and give an interpretation of sewing in terms of these exponentials of representatives of Neveu-Schwarz algebra elements. We then introduce the notion of *supergeometric vertex operator superalgebra with central charge $c \in \mathbb{C}$*. The purpose of this paper is to announce the result that the category of supergeometric vertex operator superalgebras with central charge $c \in \mathbb{C}$ is isomorphic to the category of (superalgebraic) vertex operator superalgebras with central charge $c \in \mathbb{C}$, appropriately defined.

Recall that in a vertex operator algebra, the Virasoro element $L(-1)$ plays the role of the differential operator $\frac{\partial}{\partial z}$. Thus in considering what should be the corresponding superalgebraic setting for superconformal field theory, we naturally want to consider a super-extension of the Virasoro algebra, namely the Neveu-Schwarz algebra [NS] in which the element $G(-\frac{1}{2})$ has the supercommutator $\frac{1}{2}[G(-\frac{1}{2}), G(-\frac{1}{2})] = L(-1)$. In this work, we will assume that a vertex operator superalgebra (cf. [T], [G], [FFR], [DL], and [KW]) includes the Neveu-Schwarz algebra by definition (cf. [KW]), but in addition, we extend the notion of vertex operator superalgebra to be over a Grassmann algebra instead of just $\mathbb{C}$ and to include "odd" formal variables instead of just even formal variables. This notion of *vertex operator superalgebra over a Grassmann algebra and with odd formal variables and central charge $c \in \mathbb{C}$* is in fact equivalent to the notion of vertex operator superalgebra over a Grassmann algebra without odd formal variables. However, in a vertex operator superalgebra with odd variables, the fact that $G(-\frac{1}{2})$ plays the role of the operator D (mentioned above in reference to the supergeometry) is made explicit and the correspondence with the supergeometry is more natural.

The main result we announce is that the category of supergeometric vertex operator superalgebras over a Grassmann algebra Λ_* with central charge $c \in \mathbb{C}$ and the category of vertex operator superalgebras over Λ_* with central charge $c \in \mathbb{C}$ and with (or without) odd formal variables are isomorphic. Details of the proof of this result can be found in [B].

2 Superconformal superfunctions and the Neveu-Schwarz algebra

In this section, we follow many of the conventions developed in the theory of superfunctions (cf. [D], [R]). Let Λ_∞ be the Grassmann algebra over $\mathbb{C}$ on an infinite number of generators $\zeta_1, \zeta_2, ...,$ and let $I_\infty = \{(i) = (i_1, i_2, \ldots, i_{2n}) : i_1 < i_2 < \cdots < i_{2n}, i_l \in \mathbb{Z}_+, n \in \mathbb{N}\}$, $J_\infty = \{(j) = (j_1, j_2, \ldots, j_{2n+1}) : j_1 < j_2 < \cdots < j_{2n+1}, j_l \in \mathbb{Z}_+, n \in \mathbb{N}\}$, and $K_\infty = I_\infty \cup J_\infty$. As a vector space Λ_∞ has a natural $\mathbb{Z}_2$-grading given by $\Lambda_\infty = \Lambda_\infty^0 \oplus \Lambda_\infty^1$ where $\Lambda_\infty^0 = \{a \in \Lambda_\infty : a = \sum_{(i) \in I_\infty} c_{(i)} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_{2n}}, c_{(i)} \in \mathbb{C}\}$ is the *even* subspace and $\Lambda_\infty^1 = \{a \in \Lambda_\infty : a = \sum_{(j) \in J_\infty} c_{(j)} \zeta_{j_1} \zeta_{j_2} ... \zeta_{j_{2n+1}}, c_{(j)} \in \mathbb{C}\}$ is the *odd* subspace. (Note that $\zeta_{(\emptyset)} = 1$.) We can also decompose Λ_∞ into *body* $(\Lambda_\infty)_B = \{c_{(\emptyset)} \in \mathbb{C}\}$ and *soul* $(\Lambda_\infty)_S = \{a \in \Lambda_\infty : a = \sum_{\substack{(k) \in K_\infty \\ k \neq (\emptyset)}} c_{(k)} \zeta_{k_1} \zeta_{k_2} ... \zeta_{k_n}, c_{(k)} \in \mathbb{C}\}$ subspaces such that $\Lambda_\infty = (\Lambda_\infty)_B \oplus (\Lambda_\infty)_S$. For $a \in \Lambda_\infty$, we write $a = a_B + a_S$ for its body and soul decomposition.

Let z_B be a complex variable and $h(z_B)$ analytic. For z a variable in Λ_∞^0, we define $h(z) = \sum_{n \in \mathbb{N}} \frac{z_S^n}{n!} h^{(n)}(z_B)$. Note that if $h(z_B)$ is convergent in an open neighborhood $N \subseteq \mathbb{C}$ of z_B then $h(z)$ is well-defined (i.e., convergent) in the open neighborhood $\{z = z_B + z_S \in \Lambda_\infty^0 : z_B \in N\} \subseteq \Lambda_\infty$. Let $f(z) = \sum_{(k) \in K_\infty} f_{(k)}(z) \zeta_{k_1} \zeta_{k_2} \cdots \zeta_{k_n}$ where each $f_{(k)}(z_B)$ is analytic. We say that f is a *superanalytic Λ_∞-superfunction in $(1,0)$-variables*. If $f(z) \in \Lambda_\infty^0$ (resp., Λ_∞^1) for all z in the domain of f, then f is said to be *even* (resp., *odd*). Suppose $f_{(k)}(z_B)$ is convergent in an open neighborhood $N_{(k)} \subseteq \mathbb{C}$ of z_B. If there exists an open subset $N \subseteq \bigcap_{(k) \in K_\infty} N_{(k)}$ such that $N \neq \emptyset$, then $f(z_B)$ is convergent in N, and consequently $f(z)$ is convergent in $\{z = z_B + z_S \in$

$\Lambda_\infty^0 : z_B \in N\}$.

Let $U \subseteq \Lambda_\infty$ and $H : U \longrightarrow \Lambda_\infty$, $(z, \theta) \mapsto H(z, \theta)$. We say that H is a *superanalytic* Λ_∞- *superfunction in (1,1)-variables* if H is of the form $H(z, \theta) = (f(z) + \theta\xi(z), \psi(z) + \theta g(z))$ where f, g, ξ, and ψ are superanalytic Λ_∞-superfunctions in $(1, 0)$-variables. If f, g, ξ, and ψ are convergent in the open sets N_f, N_g, N_ξ, $N_\psi \subseteq \Lambda_\infty^0$, respectively, and there exists $N_H \subseteq (N_f \cap N_g \cap N_\xi \cap N_\psi)$ such that $N_H \neq \emptyset$, then $H(z, \theta)$ is well-defined (i.e., convergent) for $\{(z, \theta) \in \Lambda_\infty : z \in N_H\}$.

Consider the topology on Λ_∞ given by the product of the usual topology on $(\Lambda_\infty)_B = \mathbb{C}$ and the trivial topology on $(\Lambda_\infty)_S$. This topology on Λ_∞ is called the *DeWitt topology*. The natural domain U of any superanalytic Λ_∞-superfunction is an open set in the DeWitt topology on Λ_∞.

Let $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$. We say that $H(z, \theta) = (\tilde{z}, \tilde{\theta})$ is *superconformal* if $D\tilde{z} = \tilde{\theta}D\tilde{\theta}$. This is equivalent to

$$H(z, \theta) = \left(f(z) + \theta\psi(z)\sqrt{f'(z) + \psi(z)\psi'(z)}, \psi(z) + \theta\sqrt{f'(z) + \psi(z)\psi'(z)} \right).$$

Thus a superconformal Λ_∞-superfunction is determined by an even Λ_∞-superfunction in $(1, 0)$-variables f, an odd Λ_∞-superfunction in $(1, 0)$-variables ψ, and a choice of square root for $f'_B(z_B)$. A square root for $f'_B(z_B)$ is called a *spin structure*. Fix a branch of the complex logarithm. For the double valued square root, we will call the structure given by $\sqrt{1} = 1$ the positive square root structure, and the structure given by $\sqrt{1} = -1$ the negative square root structure. If $H(z, \theta)$ has positive square root structure then $H(z, -\theta)$ has negative square root structure. Let $(\Lambda_\infty^0)^\times = \{a \in \Lambda_\infty^0 : (a_B) \neq 0\}$. For $a_0 \in (\Lambda_\infty^0)^\times$, define the linear operators $a_0^{z\frac{\partial}{\partial z}}$ and $a_0^{\frac{1}{2}\theta\frac{\partial}{\partial\theta}}$ on $\Lambda_\infty[z, z^{-1}, \theta]$ by $a_0^{z\frac{\partial}{\partial z}} \cdot c\theta^m z^n = c\theta^m a_0^n z^n$ and $a_0^{\frac{1}{2}\theta\frac{\partial}{\partial\theta}} \cdot c\theta^m z^n = c\sqrt{a_0}^m \theta^m z^n$ for $c \in \Lambda_\infty$, $m \in \mathbb{Z}_2$, and $n \in \mathbb{Z}$.

Let Λ_∞^∞ denote the set of infinite series in $\Lambda_\infty^0 \oplus \Lambda_\infty^1$ indexed by $j \in \mathbb{Z}_+$ in the even coordinates and $j - \frac{1}{2}$ for $j \in \mathbb{Z}_+$ in the odd coordinates. We will denote an element of Λ_∞^∞ by $(A, M) = \{(A_j, M_{j-\frac{1}{2}})\}_{j\in\mathbb{Z}_+}$ for $A_j \in \Lambda_\infty^0$ and $M_{j-\frac{1}{2}} \in \Lambda_\infty^1$. The following two propositions (proved in [B]) characterize certain superconformal superfunctions in terms of exponentials of certain superderivations. This characterization is analogous to Huang's [H1] characterization of certain conformal functions in terms of exponentials of certain derivations.

Proposition 2.1 *Let $(A, M) \in \Lambda_\infty^\infty$, and $a_0 \in (\Lambda_\infty^0)^\times$. The formal exponential of formal differential operators applied to (z, θ)*

$$H(z, \theta) = \exp\left(\sum_{j\in\mathbb{Z}_+} \left(A_j \left(z^{j+1}\frac{\partial}{\partial z} + (\frac{j+1}{2})\theta z^j\frac{\partial}{\partial\theta} \right) \right. \right. \tag{2.1}$$

$$\left. \left. + M_{j-\frac{1}{2}} z^j \left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z} \right) \right) \right) \cdot a_0^{\left(z\frac{\partial}{\partial z} + (\frac{1}{2})\theta\frac{\partial}{\partial\theta}\right)} \cdot (z, \theta)$$

is a formal power series in z and θ of the form

$$\left(a_0 z + \sum_{j\in\mathbb{Z}_+} a_j z^{j+1} + \theta \sum_{j\in\mathbb{Z}_+} n_j z^j, \sum_{j\in\mathbb{Z}_+} m_j z^j + \theta(\sqrt{a_0} + \sum_{j\in\mathbb{Z}_+} b_j z^j) \right)$$

for $a_j, b_j \in \Lambda_\infty^0$ and $m_j, n_j \in \Lambda_\infty^1$. If this formal power series converges in a (DeWitt) open neighborhood in Λ_∞, it is superconformal with positive square root structure, and any superconformal superfunction vanishing at zero with positive square root structure is of the form (2.1).

Proposition 2.2 *Let $(B, N) = \{(B_j, N_{j-\frac{1}{2}})\}_{j\in\mathbb{Z}_+}$ be an infinite series in $\Lambda_\infty^0 \oplus \Lambda_\infty^1$. The formal exponential of formal differential operators applied to $(\frac{1}{z}, \frac{i\theta}{z})$*

$$H(z,\theta) = \exp\left(-\sum_{j\in\mathbb{Z}_+}\left(B_j\left(z^{-j+1}\frac{\partial}{\partial z} + (\frac{-j+1}{2})\theta z^{-j}\frac{\partial}{\partial\theta}\right)\right.\right.$$

$$\left.\left. + N_{j-\frac{1}{2}}z^{-j+1}\left(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}\right)\right)\right)\cdot\left(\frac{1}{z}, \frac{i\theta}{z}\right) \tag{2.2}$$

is a formal power series in z^{-1} and θ of the form

$$\left(z^{-1} + \sum_{j\in\mathbb{Z}_+} a_j z^{-j-1} + \theta\sum_{j\in\mathbb{Z}_+} n_j z^{-j-1}, \sum_{j\in\mathbb{Z}_+} m_j z^{-j} + \theta(iz^{-1} + \sum_{j\in\mathbb{Z}_+} b_j z^{-j-1})\right)$$

for $a_j, b_j \in \Lambda_\infty^0$ and $m_j, n_j \in \Lambda_\infty^1$. If this formal power series converges in a (DeWitt) open neighborhood in Λ_∞, it is superconformal with positive square root structure, and any super-conformal superfunction vanishing at $z_B = \infty$ with positive square root structure is of the form (2.2).

The two propositions given above can be formulated in terms of the negative square root structure by the superconformal transformation $J(z,\theta) = (z, -\theta)$.

Consider the Neveu-Schwarz Lie superalgebra $\mathfrak{ns}$ (a certain super-extension of the Virasoro algebra) generated by a central element c, even elements $L(n)$ and odd elements $G(n + \frac{1}{2})$ for $n \in \mathbb{Z}$ with the following supercommutation relations

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c,$$

$$\left[G(m + \frac{1}{2}), L(n)\right] = (m - \frac{n-1}{2})G(m+n+\frac{1}{2})$$

$$\left[G(m + \frac{1}{2}), G(n - \frac{1}{2})\right] = 2L(m+n) + \frac{1}{3}(m^2 + m)\delta_{m+n,0}c,$$

for $m, n \in \mathbb{Z}$.

Let x be an *even* formal variable (i.e., x commutes with Λ_∞) and φ be an *odd* formal variable (i.e., φ commutes with x and Λ_∞^0, anti-commutes with Λ_∞^1, and $\varphi^2 = 0$). For any $s \in \mathbb{C}\backslash\{0\}$ and $t \in \mathbb{C}$,

$$L(n)_t = -\left(x^{n+1}\frac{\partial}{\partial x} + \left(\frac{n-1}{2} + t\right)\varphi x^n\frac{\partial}{\partial\varphi}\right)$$

$$G(n + \frac{1}{2})_{t,s} = -\left(sx^{n+t}\frac{\partial}{\partial\varphi} - \frac{1}{s}\varphi x^{n-t+2}\frac{\partial}{\partial x}\right)$$

gives a representation of $\mathfrak{ns}$ with $c = 0$. Propositions 2.1 and 2.2 state that formally $L(n)_1$ and $G(n + \frac{1}{2})_{1,1}$ for $n \in \mathbb{N}$ are the superconformal infinitesimal transformations at zero with positive square root structure, and $L(-n)_1$ and $G(-n + \frac{1}{2})_{1,1}$ for $n \in \mathbb{Z}_+$ are the superconformal infinitesimal transformations at infinity with positive square root structure. The corresponding results for the negative square root structure state that $L(n)_1$ and $G(n+\frac{1}{2})_{1,-1}(= -G(n+\frac{1}{2})_{1,1})$ for $n \in \mathbb{N}$ are the superconformal infinitesimal transformations at zero with negative square root structure, and $L(-n)_1$ and $G(-n+\frac{1}{2})_{1,-1}(= -G(-n+\frac{1}{2})_{1,1})$ for $n \in \mathbb{Z}_+$ are the superconformal infinitesimal transformations at infinity with negative square root structure.

3 Superspheres with tubes and the sewing operation

In this section, we extend Huang's [H1] definition of the moduli space of spheres with tubes and a sewing operation to a definition of the moduli space of superspheres with tubes and a sewing operation. Though it is similar in spirit, we find that in the super case there is a great deal of non-trivial additional structure involving the soul coordinates.

By *supersphere* we will mean a supermanifold with DeWitt topology over Λ_∞ (cf. [D]) such that its body is a genus-zero one-dimensional connected compact complex manifold and its transition functions are superconformal with a given square root structure. A supersphere with $1 + n$ tubes ($n \in \mathbb{N}$) is a supersphere S with 1 negatively oriented point and n positively ordered points (called *punctures*) and local superconformal coordinates vanishing at the punctures. A *superconformal equivalence F* from one supersphere S_1 with $1 + n$ tubes to another supersphere S_2 with $1 + n$ tubes which preserves square root structure is a superconformal isomorphism from the underlying supersphere of S_1 to the underlying supersphere of S_2 such that the i-th puncture of S_1 is mapped to the i-th puncture of S_2, the pull-back of the local coordinate map vanishing at the i-th puncture of S_2 is equal to the local coordinate map vanishing at the i-th puncture of S_1 in some neighborhood of this puncture, and the square root structures on S_1 and S_2 are the same.

Let S_1 be a supersphere with $1 + n$ ($n > 0$) tubes and S_2 a supersphere with $1 + m$ tubes. Let $p_0, ..., p_n$ be the punctures of S_1 with local coordinate charts (U_i, Ω_i) at p_i. Let $q_0, ..., q_n$ be the punctures of S_2 with local coordinate charts (V_i, Ξ_i) at q_i. We will describe the operation of sewing the second supersphere at q_0 to the first supersphere at p_i for some fixed $0 \leq i \leq n$, and by the uniformization theorem for super-Riemann surfaces [CR], the resulting supermanifold will be a supersphere. Assume that there exists a positive number r such that $\Omega_i(U_i)$ contains the closed set $\bar{\mathcal{B}}_0^r = \bar{B}_0^r \times (\Lambda_\infty)_S$ centered at 0 with radius r in the body, and $\Xi_0(V_0)$ contains the closed set $\bar{\mathcal{B}}_0^{1/r}$ centered at 0 with radius $1/r$ in the body. Assume also that p_i and q_0 are the only punctures in $\Omega_i^{-1}(\bar{\mathcal{B}}_0^r)$ and $\Xi_0^{-1}(\bar{\mathcal{B}}_0^{1/r})$ respectively. In this case we say that the i-th puncture of the first supersphere with tubes can be sewn with the 0-th puncture of the second supersphere with tubes. From these two superspheres with tubes, we can obtain a supersphere with $1 + (n + m - 1)$ tubes by cutting $\Omega_i^{-1}(\bar{\mathcal{B}}_0^r)$ and $\Xi_0^{-1}(\bar{\mathcal{B}}_0^{1/r})$ from S_1 and S_2 respectively, and then identifying the boundaries of the resulting surfaces using the map $\Omega_i \circ I \circ \Xi_0^{-1}$ where I is the map from $\Lambda_\infty^\times$ to itself given by $I(z, \theta) = \left(\frac{1}{z}, \frac{i\theta}{z} \right)$. The punctures (with ordering) of this supersphere with tubes are $p_0, ..., p_{i-1}, q_1, ..., q_m, p_{i+1}, ..., p_n$. The local coordinates vanishing at these punctures are (U_j, Ω_j) at p_j and (V_k, Ξ_k) at q_k. This supersphere is denoted $S_1 \; _i\infty_0 \; S_2$ (using Vafa's [V] notation for the sewing of two spheres). Note that this resulting supersphere is independent of the positive number r.

The collection of all superconformal equivalence classes of superspheres with tubes and a given square root structure is called *the moduli space of superspheres with tubes*. The global superconformal transformations with positive (resp., negative) square root structure are generated by $L(\pm 1)_1$, $L(0)_1$, and $G(\pm\frac{1}{2})_{1,1}$ (resp., $G(\pm\frac{1}{2})_{1,-1} = -G(\pm\frac{1}{2})_{1,1}$). Note that since in defining the sewing operation of two superspheres, we chose the identification of boundaries to be $I(z, \theta) = \left(\frac{1}{z}, \frac{i\theta}{z} \right)$, the sewing of two superspheres with the same square root structure results in a supersphere with the same square root structure as the original two superspheres. (If we had chosen $I^{-1}(z, \theta) = \left(\frac{1}{z}, -\frac{i\theta}{z} \right)$, the sewing of two superspheres with the same square root structure would result in a supersphere with opposite square root structure.) Thus the sewing operation described above is a well-defined (partial) operation on the moduli space of

superspheres with tubes and a given square root structure. The operation is partial since not all superspheres with tubes can be sewn together. As in [H1] and [H2], this is an important feature in that the non-sewability of certain spheres reflects information related to the analytic structure of the the moduli space. For a given branch cut in the complex plane, there are two square root structures, and thus we obtain two moduli spaces – one with a positive square root structure and one with a negative square root structure with respect to the branch cut. We can explicitly describe these moduli spaces using Propositions 2.1 and 2.2 to describe the local coordinates. Let $(\Lambda_\infty^0)^\times \times \mathcal{H}$ be the subset of $(\Lambda_\infty^0)^\times \times \Lambda_\infty^\infty$ consisting of all elements (a_0, A, M) such that (2.1) is a convergent power series in some neighborhood of zero, $\mathcal{H}^{(0)}$ the subset of Λ_∞^∞ containing all elements (B, N) such that (2.2) is a convergent power series in some neighborhood of infinity, and SM^{n-1} the subset of elements in Λ_∞^{n-1} with distinct non-zero bodies. In [B], it is shown that the moduli space of superspheres with $1 + n$ tubes ($n > 0$) and a given square root structure can be identified with the set $SK(n) = SM^{n-1} \times \mathcal{H}^{(0)} \times ((\Lambda_\infty^0)^\times \times \mathcal{H})^n$, and the moduli space of superspheres with one tube and a given square root structure can be identified with the set $SK(0) = \{(B, N) \in \mathcal{H}^{(0)} : (A_1, M_{\frac{1}{2}}) = (0, 0)\}$. Thus we will refer to SK^+ (resp., SK^-) as the moduli space of superspheres with tubes and positive (resp., negative) square root structure. Although as sets $SK^+ = SK^-$, the underlying geometric structures they represent are distinct, and we can map one to the other via the transformation $J(z, \theta) = (z, -\theta)$ on the underlying superspheres with tubes. The definition of sewing of superspheres gives a (partial) operation on SK^+ (resp., SK^-) which we again denote by $_i\infty_0$, and as sets, the sewing operation on SK^+ is the same as that on SK^-. We write an element of $SK^+(n)$ or $SK^-(n)$ as $((z_1, \theta_1), ..., (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a_0^{(1)}, A^{(1)}, M^{(1)}), ..., (a_0^{(n)}, A^{(n)}, M^{(n)}))$. (When we want to refer to SK^+ or SK^-, we will write $SK^\pm(n)$.) The symmetric group on $n - 1$ letters S_{n-1} acts on $SK^\pm(n)$ by permuting the (z_i, θ_i) and $(a_0^{(i)}, A^{(i)}, M^{(i)})$ for $i = 1, ..., n - 1$. In [B], we extend this action to an action of S_n on $SK^\pm(n)$. We will denote by $\mathbf{0}$ the element of Λ_∞^∞ with all components equal to 0. The superconformal transformation $J(z, \theta) = (z, -\theta)$ induces a map (which we also denote by J) from $SK^\pm$ to $SK^\mp$ given by

$$J((z_1, \theta_1), ..., (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a_0^{(1)}, A^{(1)}, M^{(1)}), ..., (a_0^{(n)}, A^{(n)}, M^{(n)})) =$$

$$((z_1, -\theta_1), ..., (z_{n-1}, -\theta_{n-1}); (A^{(0)}, M^{(0)}), (a_0^{(1)}, A^{(1)}, M^{(1)}), ..., (a_0^{(n)}, A^{(n)}, M^{(n)})).$$

Remark 3.1 *The moduli space $SK^\pm$ of superspheres with tubes and a given positive or negative square root structure and the sewing operation on $SK^\pm$ is a partial operad (cf. [HL]).*

Let $Q_1 \in SK^\pm(n)$ for $n \in \mathbb{Z}_+$, and $Q_2 \in SK^\pm(m)$ for $m \in \mathbb{N}$. Let the coordinates at the puncture (z_i, θ_i) of Q_1 be $H(z - z_i - \theta\theta_i, \theta - \theta_i)$ where $H(z, \theta)$ is given by (2.1), and let the local coordinates at infinity of Q_2 be given by (2.2). The following proposition describes the change of local coordinates of the resulting supersphere Q_1 $_i\infty_0$ Q_2, and this description is given in terms of elements of the Neveu-Schwarz algebra with central charge $c \in \mathbb{C}$.

Proposition 3.2 *Let $(\mathcal{A}, \mathcal{M}) = \{(\mathcal{A}_j, \mathcal{M}_{j-\frac{1}{2}})\}_{j \in \mathbb{Z}_+}$ and $(\mathcal{B}, \mathcal{N}) = \{(\mathcal{B}_j, \mathcal{N}_{j-\frac{1}{2}})\}_{j \in \mathbb{Z}_+}$ be two sequences of formal variables, $\mathcal{A}_j$ and $\mathcal{B}_j$ even and $\mathcal{M}_{j-\frac{1}{2}}$ and $\mathcal{N}_{j-\frac{1}{2}}$ odd, let α_0 be another even formal variable, and let V be a positive energy module for the Neveu-Schwarz algebra. There exist unique canonical series $(\Psi_j, \Psi_{j-\frac{1}{2}}) = (\Psi_j, \Psi_{j-\frac{1}{2}})(\alpha_0, \mathcal{A}, \mathcal{M}, \mathcal{B}, \mathcal{N})$ for $j \in \mathbb{Z}$, and $\Gamma = \Gamma(\alpha_0, \mathcal{A}, \mathcal{M}, \mathcal{B}, \mathcal{N})$ in $\mathbb{C}[\alpha_0, \alpha_0^{-1}, \sqrt{\alpha_0}, \sqrt{\alpha_0}^{-1}][[\mathcal{A}, \mathcal{M}, \mathcal{B}, \mathcal{N}]]$ such that*

$$e^{-\sum_{j \in \mathbb{Z}_+} (\mathcal{A}_j L(j) + \mathcal{M}_{j-\frac{1}{2}} G(j-\frac{1}{2}))} \; \alpha_0^{-L(0)} \; e^{-\sum_{j \in \mathbb{Z}_+} (\mathcal{B}_j L(-j) + \mathcal{N}_{j-\frac{1}{2}} G(-j+\frac{1}{2}))} =$$

$$e^{\sum_{j \in \mathbb{Z}_+} (\Psi_{-j} L(-j) + \Psi_{-j+\frac{1}{2}} G(-j+\frac{1}{2}))} \; e^{\sum_{j \in \mathbb{Z}_+} (\Psi_j L(j) + \Psi_{j-\frac{1}{2}} G(j-\frac{1}{2}))} \; e^{\Psi_0 L(0)} \; \alpha_0^{-L(0)} \; e^{\Gamma c}$$

as operators in $(\mathrm{End}\, V)[\alpha_0, \alpha_0^{-1}, \sqrt{\alpha_0}, \sqrt{\alpha_0}^{-1}][[\mathcal{A}, \mathcal{M}, \mathcal{B}, \mathcal{N}]]$.

The series Γ can easily be calculated up to second order terms in the $\mathcal{A}_j$'s, $\mathcal{M}_{j-\frac{1}{2}}$'s, $\mathcal{B}_j$'s, and $\mathcal{N}_{j-\frac{1}{2}}$'s for $j \in \mathbb{Z}_+$. In fact,

$$
\begin{aligned}
\Gamma &= \Gamma(\alpha_0, \mathcal{A}, \mathcal{M}, \mathcal{B}, \mathcal{N}) \\
&= \sum_{j \in \mathbb{Z}_+} \left(\left(\frac{j^3 - j}{12} \right) \alpha_0^{-j} \mathcal{A}_j \mathcal{B}_j + \left(\frac{j^2 - j}{3} \right) \sqrt{\alpha_0} \alpha_0^{-j} \mathcal{N}_{j-\frac{1}{2}} \mathcal{M}_{j-\frac{1}{2}} \right) + \Gamma_0
\end{aligned}
$$

where Γ_0 contains only terms with products of at least three of the $\mathcal{A}_j$'s, $\mathcal{M}_{j-\frac{1}{2}}$'s, $\mathcal{B}_j$'s, and $\mathcal{N}_{j-\frac{1}{2}}$'s for $j \in \mathbb{Z}_+$ (but not all of the three $\mathcal{A}_j$'s, $\mathcal{M}_{j-\frac{1}{2}}$'s, $\mathcal{B}_j$'s, or $\mathcal{N}_{j-\frac{1}{2}}$'s). The series Γ has the following convergence property:

Proposition 3.3 *Let*

$$Q_1^\pm = ((z_1, \theta_1), ..., (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a_0^{(1)}, A^{(1)}, M^{(1)}), ..., (a_0^{(n)}, A^{(n)}, M^{(n)})) \in SK^\pm(n),$$

$$Q_2^\pm = ((z_1, \theta_1), ..., (z_{m-1}, \theta_{m-1}); (B^{(0)}, N^{(0)}), (b_0^{(1)}, B^{(1)}, N^{(1)}), ..., (b_0^{(m)}, B^{(m)}, N^{(m)})) \in SK^\pm(m).$$

If the i-th tube of $Q_1^\pm$ can be sewn with the 0-th tube of $Q_2^\pm$, the t-series

$$e^{\Gamma(t^{-1} a_0^{(i)}, A^{(i)}, M^{(i)}, B^{(0)}, N^{(0)}) c}$$

is absolutely convergent at $t = 1$.

4 The linear algebra of $\mathbf{Z_2}$-graded Λ_∞-modules with $\frac{1}{2}\mathbf{Z}$-graded finite-dimensional weight spaces

For any $\mathbb{Z}_2$-graded vector space $V = V^0 \oplus V^1$, define the *sign* of v homogeneous in V to be $\eta(v) = i$ for $v \in V^i$, $i \in \mathbb{Z}_2$. Let

$$V = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^0 \oplus \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^1 = V^0 \oplus V^1$$

with

$$\dim V_{(n)} < \infty \quad \text{for} \quad n \in \frac{1}{2}\mathbb{Z},$$

be a $\frac{1}{2}\mathbb{Z}$-graded (by weight) Λ_∞-module with finite-dimensional homogeneous weight spaces $V_{(n)}$ which is also $\mathbb{Z}_2$-graded (by sign). Let

$$V' = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^*$$

be the graded dual space of V,

$$\bar{V} = \prod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} = V'^*$$

the algebraic completion of V, and $\langle \cdot, \cdot \rangle$ the natural pairing between V' and $\bar{V}$. For any $n \in \mathbb{N}$, let

$$\mathcal{SF}_V(n) = \text{Hom}_{\Lambda_\infty}(V^{\otimes n}, \bar{V}).$$

For any $m \in \mathbb{Z}_+$, $n \in \mathbb{N}$, and any positive integer $i \leq m$, we define the *t-contraction*

$$
\begin{aligned}
{}_i*_0 : \mathcal{SF}_V(m) \times \mathcal{SF}_V(n) &\rightarrow \text{Hom}(V^{\otimes(m+n-1)}, V[[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]]) \\
(f, g) &\mapsto (f \; _i*_0 \; g)_t,
\end{aligned}
$$

by

$$(f \; _i*_0 \; g)_t(v_1 \otimes \cdots \otimes v_{n+m-1}) =$$

$$\sum_{k \in \frac{1}{2}\mathbb{Z}} f(v_1 \otimes \cdots \otimes v_{i-1} \otimes P_k(g(v_i \otimes \cdots \otimes v_{i+n-1})) \otimes v_{i+n} \otimes \cdots \otimes v_{m+n-1})t^k \qquad (4.3)$$

for all $v_i, ..., v_{m+n-1} \in V$, where for any $k \in \frac{1}{2}\mathbb{Z}$, $P_k : V \rightarrow V_{(k)}$ is the projection map. If we want to substitute complex values for t into equation (4.3), we must choose a square root. However, we have already fixed a branch cut and defined the positive and negative single-valued square roots for this branch cut. We denote the two corresponding positive and negative t-contractions by $(f \; _i*_0 \; g)_t^+$ and $(f \; _i*_0 \; g)_t^-$, respectively, for $t \in \mathbb{C}$.

If for arbitrary $v' \in V'$, $v_1, ..., v_{m+n-1} \in V$, the formal Laurent series in $t^{\frac{1}{2}}$

$$\langle v', (f \; _i*_0 \; g)_t^{\pm}(v_1 \otimes \cdots \otimes v_{n+m-1}) \rangle$$

is absolutely convergent when $t = 1$, then $(f \; _i*_0 \; g)_1^{\pm}$ is well-defined as an element of $\mathcal{SF}_V(m + n - 1)$, and we define the *positive (resp., negative) contraction* $(f \; _i*_0 \; g)^+$, (resp. $(f \; _i*_0 \; g)^-$) in $\mathcal{SF}_V(m + n - 1)$ of f and g by

$$(f \; _i*_0 \; g)^{\pm} = (f \; _i*_0 \; g)_1^{\pm}.$$

Let $(l \; k) \in S_n$ be the permutation on n letters which switches the l-th and k-th letters, for $l, k = 1, ..., n$, $l < k$. We define an action of the transposition $(l \; k)$ on $V^{\otimes n}$ by

$$(l \; k)(v_1 \otimes \cdots \otimes v_l \otimes \cdots \otimes v_k \otimes \cdots \otimes v_n) = (-1)^{\eta(l \; k)}(v_1 \otimes \cdots \otimes v_k \otimes \cdots \otimes v_l \otimes \cdots \otimes v_n)$$

for v_j of homogeneous sign in V, where

$$\eta(l \; k) = \sum_{j=l+1}^{k-1} \eta(v_j)(\eta(v_l) + \eta(v_k)) + \eta(v_k)\eta(v_l).$$

Let $\sigma \in S_n$ be a permutation on n letters. Then σ is the product of transpositions $\sigma = \sigma_1 \cdots \sigma_m$, $\sigma_i = (l_i \; k_i)$, $l_i, k_i \in \{1, ..., n\}$, $l_i < k_i$, $i = 1, ..., m$. Thus we have an action of S_n on $V^{\otimes n}$ given by

$$\sigma(v_1 \otimes \cdots \otimes v_n) = \sigma_1 \cdots \sigma_m(v_1 \otimes \cdots \otimes v_n) = (-1)^{\eta(\sigma_1)+\cdots+\eta(\sigma_m)}v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

This action of S_n induces a left action of S_n on $\mathcal{SF}_V(n)$ given by

$$\sigma(f)(v_1 \otimes \cdots \otimes v_n) = f(\sigma^{-1}(v_1 \otimes \cdots \otimes v_n)),$$

for $f \in \mathcal{SF}_V(n)$.

Since V is a $\mathbb{Z}_2$-graded Λ_∞-module, End V has a natural $\mathbb{Z}_2$-grading given by even operators $(\text{End } V)^0 = \{P \in \text{End } V : PV^m \subset V^m \text{ for } m \in \mathbb{Z}_2\}$ and odd operators $(\text{End } V)^1 = \{P \in \text{End } V : PV^m \subset V^{(m+1) \bmod 2} \text{ for } m \in \mathbb{Z}_2\}$. Also, End V has a natural Lie superalgebra structure with supercommutator given by $[P_1, P_2] = P_1 P_2 - (-1)^{\eta(P_1)\eta(P_2)} P_2 P_1$, for P_1 and P_2 of homogeneous sign in End V.

If $P \in \text{End } V$, the corresponding adjoint operator on V', if it exists, is denoted by P'. The condition for the existence of P' is that the linear functional on V defined by the right-hand side of

$$\langle P'v', v \rangle = \langle v', Pv \rangle, \text{ for } v \in V, \ v' \in V'$$

should lie in V'. If there exists $n \in \frac{1}{2}\mathbb{Z}$ such that P maps $V_{(k)}$ to $V_{(n+k)}$ for any $k \in \frac{1}{2}\mathbb{Z}$, we say that P has *weight* n. It is easy to see that P has weight n if and only if its adjoint P' exists and has weight $-n$ as an operator on V', and that P is even (resp., odd) if and only if its adjoint exists and is even (resp., odd). In the case that V is a module for the Neveu-Schwarz algebra graded by the eigenvalues of $L(0)$, the adjoint operator $L'(-n)$ for $n \in \mathbb{Z}$ corresponding to $L(-n)$ exists and is even with weight n, and the adjoint operator $G'(-n - \frac{1}{2})$ for $n \in \mathbb{Z}$ corresponding to $G(-n - \frac{1}{2})$ exists and is odd with weight $n + \frac{1}{2}$.

5 Supergeometric vertex operator superalgebras

In the definition of supergeometric vertex operator superalgebras, we need the following notion of supermeromorphic superfunction on $SK^\pm(n)$. A *supermeromorphic superfunction on $SK^\pm(n)$* $(n \in \mathbb{Z}_+)$ is a superfunction $F : SK^\pm(n) \to \Lambda_\infty$ of the form

$$F((z_1, \theta_1), ..., (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a_0^{(1)}, A^{(1)}, M^{(1)}), ..., (a_0^{(n)}, A^{(n)}, M^{(n)}))$$

$$= \frac{1}{\prod_{i=1}^{n-1} z_i^{s_i} \prod_{1 \leq i < j \leq n-1} (z_i - z_j - \theta_i \theta_j)^{s_{ij}}} F_0((z_1, \theta_1), ..., (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), \tag{5.4}$$

$$(a_0^{(1)}, A^{(1)}, M^{(1)}), ..., (a_0^{(n)}, A^{(n)}, M^{(n)}))$$

where s_i and s_{ij} are nonnegative integers and

$$F_0((z_1, \theta_1), ..., (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a_0^{(1)}, A^{(1)}, M^{(1)}), ..., (a_0^{(n)}, A^{(n)}, M^{(n)}))$$

is a polynomial in the z_i's, θ_i's, $a_0^{(i)}$'s, $(a_0^{(i)})^{-1}$'s, $A_j^{(i)}$'s, and $M_{j-\frac{1}{2}}^{(i)}$'s. For $n = 0$ a *supermeromorphic superfunction on $SK^\pm(0)$* is a polynomial in the components of elements of $SK^\pm(0)$.

For $L \in \mathbb{N}$, let Λ_L, be the Grassmann subalgebra over $\mathbb{C}$ of Λ_∞ on generators $\zeta_1, \zeta_2, ..., \zeta_L$. We use the notation Λ_* to denote Λ_L for some $L \in \mathbb{N}$ or Λ_∞. Recall that a formal variable is even if it commutes with Λ_∞ and all other formal variables and is odd if it commutes with Λ_∞^0, and anti-commutes with Λ_∞^1 and all odd formal variables including itself, i.e., its square is zero.

Definition 5.1 A *(N = 1 Neveu-Schwarz) supergeometric vertex operator superalgebra over* Λ_* *with positive square root structure* is a $\frac{1}{2}\mathbb{Z}$-graded (by weight) Λ_∞-module which is also $\mathbb{Z}_2$-graded (by sign)

$$V = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^0 \oplus \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^1 = V^0 \oplus V^1$$

such that only the subspace Λ_* of Λ_∞ acts non-trivially on V,

$$\dim V_{(n)} < \infty \quad \text{for} \quad n \in \frac{1}{2}\mathbb{Z},$$

and for any $n \in \mathbb{N}$, a map

$$\nu_n^+ : SK^+(n) \to S\mathcal{F}_V(n)$$

satisfying the following axioms:

(1) *Positive energy axiom:*

$$V_{(n)} = 0 \quad \text{for } n \text{ sufficiently small.}$$

(2) *Grading axiom:* Let $v' \in V'$, $v \in V_{(n)}$, and $a_0 \in (\Lambda_\infty^0)^\times$. Then

$$\langle v', \nu_1^+(\mathbf{0}, (a_0, \mathbf{0}))(v) \rangle = a_0^{-n} \langle v', v \rangle.$$

(3) *Supermeromorphicity axiom:* For any $n \in \mathbb{Z}_+$, $v' \in V'$, and $v_1, ..., v_n \in V$, the function

$$Q \mapsto \langle v', \nu_n^+(Q)(v_1 \otimes \cdots \otimes v_n) \rangle$$

on $SK^+(n)$ is a canonical supermeromorphic superfunction (in the sense of (5.4)), and if (z_i, θ_i) and (z_j, θ_j) are the i-th and j-th punctures of $Q \in SK^+(n)$ respectively $(i, j \in \{1, ..., n\}, i \neq j)$, then for any v_i and v_j in V there exists $N(v_i, v_j) \in \mathbb{Z}_+$ such that for any $v' \in V'$ and $v_k \in V$, $k \neq i, j$, the order of the pole $(z_i, \theta_i) = (z_j, \theta_j)$ of $\langle v', \nu_n(Q)(v_1 \otimes \cdots \otimes v_n) \rangle$ is less then $N(v_i, v_j)$.

(4) *Permutation axiom:* Let $\sigma \in S_n$. Then for any $Q \in SK^+(n)$

$$\sigma(\nu_n^+(Q)) = \nu_n^+(\sigma(Q)).$$

(5) *Sewing axiom:* There exists a unique complex number c (the *central charge* or *rank*) such that if

$$Q_1 = ((z_1, \theta_1), ..., (z_{n-1}, \theta_{n-1}); (A^{(0)}, M^{(0)}), (a_0^{(1)}, A^{(1)}, M^{(1)}), ..., (a_0^{(n)}, A^{(n)}, M^{(n)})) \in SK^+(n),$$

$$Q_2 = ((z_1', \theta_1'), ..., (z_{m-1}', \theta_{m-1}'); (B^{(0)}, N^{(0)}), (b_0^{(1)}, B^{(1)}, N^{(1)}), ..., (b_0^{(m)}, B^{(m)}, N^{(m)})) \in SK^+(m),$$

and if the i-th tube of Q_1 $(1 \leq i \leq n)$ can be sewn with the 0-th tube of Q_2, then for any $v' \in V'$, $v_1, ..., v_{n+m-1} \in V$,

$$\langle v', (\nu_n^+(Q_1) \; _i *_0 \; \nu_m^+(Q_2))_t^+(v_1 \otimes \cdots \otimes v_{n+m-1}) \rangle$$

is absolutely convergent when $t = 1$, and

$$\nu_{n+m-1}^+(Q_1 \; _i \infty_0 \; Q_2) = (\nu_n^+(Q_1) \; _i *_0 \; \nu_m^+(Q_m))^+ \; e^{-\Gamma(a_0^{(i)}, A^{(i)}, M^{(i)}, B^{(0)}, N^{(0)})c}.$$

We denote the supergeometric vertex operator superalgebra defined above by $(V, \nu^+ = \{\nu_n^+\}_{n \in \mathbb{N}})$. Replacing SK^+ by SK^- in the above definition, we have the corresponding notion of a supergeometric vertex operator superalgebra with negative square root structure (V, ν^-) where $\nu^- = \nu^+ \circ J$.

Let $(V_1, \nu^\pm)$ and $(V_2, \mu^\pm)$ be two supergeometric vertex operator superalgebras over Λ_*, $\gamma : V_1 \to V_2$ a doubly graded linear homomorphism, i.e., $\gamma : (V_1)_{(n)}^i \to (V_2)_{(n)}^i$ for $n \in \frac{1}{2}\mathbb{Z}$, and $i \in \mathbb{Z}_2$, and let $\bar{\gamma} : \bar{V}_1 \to \bar{V}_2$ be the unique extension of γ. If for any $n \in \mathbb{N}$ and any $Q \in SK^\pm(n)$

$$\bar{\gamma} \circ \nu^\pm(Q) = \mu^\pm(Q) \circ \gamma^{\otimes n},$$

we say that γ is a *homomorphism* from $(V_1, \nu^\pm)$ to $(V_2, \mu^\pm)$.

Let c be a complex number, and let $\mathbf{SG}^+(\mathbf{c}, *)$ be the category of supergeometric vertex operator superalgebras over Λ_* with positive square root structure, and $\mathbf{SG}^-(\mathbf{c}, *)$ be the category of supergeometric vertex operator superalgebras over Λ_* with negative square root structure.

Proposition 5.2 *The two categories $\mathbf{SG}^+(\mathbf{c}, *)$ and $\mathbf{SG}^-(\mathbf{c}, *)$ are isomorphic.*

Proof: We define $J^+ : \mathbf{SG}^+(\mathbf{c}, *) \to \mathbf{SG}^-(\mathbf{c}, *)$ and $J^- : \mathbf{SG}^-(\mathbf{c}, *) \to \mathbf{SG}^+(\mathbf{c}, *)$ by

$$J^+(V, \nu^+) = (V, \nu^- = \nu^+ \circ J) \quad \text{and} \quad J^+(\gamma) = \gamma$$
$$J^-(V, \nu^-) = (V, \nu^+ = \nu^- \circ J) \quad \text{and} \quad J^-(\gamma) = \gamma.$$

It is easy to see that J^+ and J^- are functors and that $J^+ \circ J^- = 1_{\mathbf{SG}^-(\mathbf{c}, *)}$ and $J^- \circ J^+ = 1_{\mathbf{SG}^+(\mathbf{c}, *)}$. ∎

6 (Superalgebraic) vertex operator superalgebras

In this section, we extend the notion of vertex operator superalgebra to include odd formal variables. In [FLM], the formal δ-function $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$ is a fundamental ingredient in the formal calculus underlying the theory of vertex operator algebras. Extending the formal calculus to include odd formal variables, we note that for any formal Laurent series $f(x) \in \Lambda_\infty[[x, x^{-1}]]$ in the even formal variable x, and for any odd formal variables φ_1 and φ_2, we have $f(x + \varphi_1 \varphi_2) = f(x) + \varphi_1 \varphi_2 f'(x)$. Thus we have the following δ-function involving three even variables and two odd variables:

$$
\begin{aligned}
\delta\left(\frac{x_1 - x_2 - \varphi_1\varphi_2}{x_0}\right) &= \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2 - \varphi_1\varphi_2)^n}{x_0^n} = \sum_{n \in \mathbb{Z}} \left(\frac{(x_1 - x_2)^n}{x_0^n} - n\varphi_1\varphi_2 \frac{(x_1 - x_2)^{n-1}}{x_0^n}\right) \\
&= \delta\left(\frac{x_1 - x_2}{x_0}\right) - \varphi_1\varphi_2 x_0^{-1} \delta'\left(\frac{x_1 - x_2}{x_0}\right).
\end{aligned}
$$

Definition 6.1 A *(N = 1 Neveu-Schwarz) vertex operator superalgebra over Λ_* and with odd variables* is a $\frac{1}{2}\mathbb{Z}$-graded (by weight) Λ_∞-module which is also $\mathbb{Z}_2$-graded (by sign)

$$V = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^0 \oplus \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^1 = V^0 \oplus V^1$$

such that only the subspace Λ_* of Λ_∞ acts non-trivially on V, and

$$\dim V_{(n)} < \infty \quad \text{for} \quad n \in \frac{1}{2}\mathbb{Z},$$

$$V_{(n)} = 0 \quad \text{for } n \text{ sufficiently small,}$$

equipped with a linear map $V \otimes V \to V[[x, x^{-1}]] \oplus \varphi V[[x, x^{-1}]]$, or equivalently,

$$
\begin{aligned}
V & \to (\text{End } V)[[x, x^{-1}]] \oplus \varphi(\text{End } V)[[x, x^{-1}]] \\
v & \mapsto Y(v, (x, \varphi)) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} + \varphi \sum_{n \in \mathbb{Z}} v_{n-\frac{1}{2}} x^{-n-1}
\end{aligned}
$$

where $v_n \in (\text{End } V)^{\eta(v)}$ and $v_{n-\frac{1}{2}} \in (\text{End } V)^{(\eta(v)+1) \bmod 2}$ for v of homogeneous sign in V, x is an even formal variable, and φ is an odd formal variable, and where $Y(v, (x, \varphi))$ denotes the *vertex operator associated with v*, and equipped also with two distinguished homogeneous vectors $\mathbf{1} \in V_{(0)}^0$ (the *vacuum*) and $\tau \in V_{(\frac{3}{2})}^1$. The following conditions are assumed for $u, v \in V$:

$$u_n v = 0 \quad \text{for } n \in \tfrac{1}{2}\mathbb{Z} \text{ sufficiently large;}$$

$$Y(\mathbf{1}, (x, \varphi)) = 1 \quad (1 \text{ on the right being the identity operator);}$$

the *creation property* holds:

$$Y(v, (x, \varphi))\mathbf{1} \in V[[x]] \oplus \varphi V[[x]] \qquad \text{and} \qquad \lim_{(x,\varphi) \to 0} Y(v, (x, \varphi))\mathbf{1} = v;$$

the *Jacobi identity* holds:

$$x_0^{-1} \delta\left(\frac{x_1 - x_2 - \varphi_1 \varphi_2}{x_0}\right) Y(u, (x_1, \varphi_1)) Y(v, (x_2, \varphi_2))$$

$$-(-1)^{\eta(u)\eta(v)} x_0^{-1} \delta\left(\frac{x_2 - x_1 + \varphi_1 \varphi_2}{-x_0}\right) Y(v, (x_2, \varphi_2)) Y(u, (x_1, \varphi_1))$$

$$= x_2^{-1} \delta\left(\frac{x_1 - x_0 - \varphi_1 \varphi_2}{x_2}\right) Y(Y(u, (x_0, \varphi_1 - \varphi_2))v, (x_2, \varphi_2));$$

the Neveu-Schwarz algebra relations hold:

$$
\begin{aligned}
[L(m), L(n)] & = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0} c, \\
\left[G(m + \tfrac{1}{2}), L(n)\right] & = (m - \frac{n-1}{2})G(m + n + \tfrac{1}{2}), \\
\left[G(m + \tfrac{1}{2}), G(n - \tfrac{1}{2})\right] & = 2L(m + n) + \frac{1}{3}(m^2 + m)\delta_{m+n,0} c,
\end{aligned}
$$

for $m, n \in \mathbb{Z}$, where

$$G(n + \frac{1}{2}) = \tau_{n+1}, \qquad \text{and} \qquad 2L(n) = \tau_{n+\frac{1}{2}} \qquad \text{for } n \in \mathbb{Z},$$

i.e.

$$Y(\tau, (x, \varphi)) = \sum_{n \in \mathbb{Z}} G(n + \frac{1}{2}) x^{-n - \frac{1}{2} - \frac{3}{2}} + 2\varphi \sum_{n \in \mathbb{Z}} L(n) x^{-n-2},$$

and $c \in \mathbb{C}$;

$$L(0)v = nv \quad \text{for} \quad n \in \frac{1}{2}\mathbb{Z} \quad \text{and} \quad v \in V_{(n)};$$

$$\left(\frac{\partial}{\partial \varphi} + \varphi \frac{\partial}{\partial x}\right) Y(v, (x, \varphi)) = Y(G(-\frac{1}{2})v, (x, \varphi)).$$

The superalgebraic vertex operator superalgebra just defined is denoted by $(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau)$.

Two consequences of the definition are that

$$Y(v, (x, \varphi)) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} + \varphi \sum_{n \in \mathbb{Z}} [G(-\tfrac{1}{2}), v_n] x^{-n-1}, \tag{6.5}$$

i.e. $v_{n-\frac{1}{2}} = [G(-\frac{1}{2}), v_n]$, and

$$\frac{\partial}{\partial x} Y(v, (x, \varphi)) = Y(L(-1)v, (x, \varphi)). \tag{6.6}$$

Let $(V_1, Y_1(\cdot, (x, \varphi)), \mathbf{1}_1, \tau_1)$ and $(V_2, Y_2(\cdot, (x, \varphi)), \mathbf{1}_2, \tau_2)$ be two vertex operator superalgebras over Λ_*. A *homomorphism* of vertex operator superalgebras with odd formal variables is a doubly graded Λ_*-module homomorphism $\gamma : V_1 \to V_2$ such that

$$\gamma(Y_1(u, (x, \varphi))v) = Y_2(\gamma(u), (x, \varphi))\gamma(v) \quad \text{for} \quad u, v \in V_1,$$

$\gamma(\mathbf{1}_1) = \mathbf{1}_2$, and $\gamma(\tau_1) = \tau_2$.

Definition 6.2 A *($N = 1$ Neveu-Schwarz) vertex operator superalgebra over Λ_* and without odd variables* is a $\frac{1}{2}\mathbb{Z}$-graded (by weight) Λ_∞-module which is also $\mathbb{Z}_2$-graded (by sign)

$$V = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^0 \oplus \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)}^1 = V^0 \oplus V^1$$

such that only the subspace Λ_* of Λ_∞ acts non-trivially on V, and

$$\dim V_{(n)} < \infty \quad \text{for} \quad n \in \frac{1}{2}\mathbb{Z},$$

$$V_{(n)} = 0 \quad \text{for } n \text{ sufficiently small},$$

equipped with a linear map $V \otimes V \to V[[x, x^{-1}]]$, or equivalently,

$$\begin{aligned} V &\to (\text{End } V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \end{aligned}$$

where $v_n \in (\text{End } V)^{\eta(v)}$ for v of homogeneous sign in V, x is an even formal variable, and where $Y(v, x)$ denotes the *vertex operator associated with v*, and equipped also with two distinguished homogeneous vectors $\mathbf{1} \in V_{(0)}^0$ (the *vacuum*) and $\tau \in V_{(\frac{3}{2})}^1$. The following conditions are assumed for $u, v \in V$:

$$u_n v = 0 \quad \text{for } n \in \tfrac{1}{2}\mathbb{Z} \text{ sufficiently large};$$

$$Y(\mathbf{1}, x) = 1 \quad (1 \text{ on the right being the identity operator});$$

the *creation property* holds:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v;$$

the *Jacobi identity* holds:

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-(-1)^{\eta(u)\eta(v)}x_0^{-1}\delta\left(\frac{-x_2+x_1}{x_0}\right)Y(v,x_2)Y(u,x_1)$$

$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2);$$

the Neveu-Schwarz algebra relations hold:

$$[L(m),L(n)] = (m-n)L(m+n)+\frac{1}{12}(m^3-m)\delta_{m+n,0}c,$$

$$\left[G(m+\frac{1}{2}),L(n)\right] = (m-\frac{n-1}{2})G(m+n+\frac{1}{2}),$$

$$\left[G(m+\frac{1}{2}),G(n-\frac{1}{2})\right] = 2L(m+n)+\frac{1}{3}(m^2+m)\delta_{m+n,0}c,$$

for $m,n\in\mathbb{Z}$, where

$$G(n+\frac{1}{2})=\tau_{n+1}\qquad\text{for }n\in\mathbb{Z},i.e.\quad Y(\tau,x)=\sum_{n\in\mathbb{Z}}G(n+\frac{1}{2})x^{-n-\frac{1}{2}-\frac{3}{2}},$$

and $c\in\mathbb{C}$;

$$L(0)v=nv\quad\text{for}\quad n\in\frac{1}{2}\mathbb{Z}\quad\text{and}\quad v\in V_{(n)};$$

$$\frac{\partial}{\partial x}Y(v,x)=Y(L(-1)v,x).$$

The superalgebraic vertex operator superalgebra just defined is denoted by $(V,Y(\cdot,x),\mathbf{1},\tau)$.

A consequence of the definition is that

$$[G(-\frac{1}{2}),Y(v,x)]=Y(G(-\frac{1}{2})v,x).\tag{6.7}$$

Note that our definition of vertex operator superalgebra (without formal variables) is an extension of the usual notion of vertex operator superalgebra (cf. [T], [DL], and [KW]) in that V is a Λ_∞-module instead of just a vector space over $\mathbb{C}$.

Let $(V_1,Y_1(\cdot,x),\mathbf{1}_1,\tau_1)$ and $(V_2,Y_2(\cdot,x),\mathbf{1}_2,\tau_2)$ be two vertex operator superalgebras over Λ_*. A *homomorphism* of vertex operator superalgebras without odd formal variables is a doubly graded Λ_*-module homomorphism $\gamma:V_1\to V_2$ such that

$$\gamma(Y_1(u,x)v)=Y_2(\gamma(u),x)\gamma(v)\quad\text{for}\quad u,v\in V_1,$$

$\gamma(\mathbf{1}_1)=\mathbf{1}_2$, and $\gamma(\tau_1)=\tau_2$.

Let c be a complex number, $\mathbf{SV_1(c},*)$ be the category of vertex operator superalgebras over Λ_* with odd formal variables, and $\mathbf{SV_2(c},*)$ be the category of vertex operator superalgebras over Λ_* without odd formal variables.

Proposition 6.3 *For any $c\in\mathbb{C}$, the two categories $\mathbf{SV_1(c},*)$ and $\mathbf{SV_2(c},*)$ are isomorphic.*

Sketch of proof: We define $F_1:\mathbf{SV_1(c},*)\to\mathbf{SV_2(c},*)$ and $F_2:\mathbf{SV_2(c},*)\to\mathbf{SV_1(c},*)$ by

$$F_1(V,Y(\cdot,(x,\varphi)),\mathbf{1},\tau)=(V,Y(\cdot,(x,0)),\mathbf{1},\tau),\quad\text{and}\quad F_1(\gamma)=\gamma$$
$$F_2(V,Y(\cdot,x),\mathbf{1},\tau)=(V,\tilde{Y}(\cdot,(x,\varphi)),\mathbf{1},\tau),\quad\text{and}\quad F_1(\gamma)=\gamma$$

where $\tilde{Y}(v,(x,\varphi))=Y(v,x)+\varphi Y(G(-\frac{1}{2})v,x)$. Using the consequences (6.5) and (6.6) of the definition of $(V,Y(\cdot,(x,\varphi)),\mathbf{1},\tau)$ and the consequence (6.7) of the definition of $(V,Y(\cdot,x),\mathbf{1},\tau)$, it is easy to see that $(V,Y(\cdot,(x,0)),\mathbf{1},\tau)$ and $(V,\tilde{Y}(\cdot,(x,\varphi)),\mathbf{1},\tau)$ are vertex operator superalgebras without and with odd variables, respectively. Then clearly F_1 and F_2 are functors, and we have $F_1\circ F_2=1_{\mathbf{SV_2(c)}}$ and $F_2\circ F_1=1_{\mathbf{SV_1(c)}}$. $\blacksquare$

7 The isomorphism between the category of supergeometric vertex operator superalgebras and the category of vertex operator superalgebras

The main result in [H2] states that the category of geometric vertex operator algebras and the category of vertex operator algebras are isomorphic. The main result that we announce is the following analogous result in the super case.

Theorem 7.1 *For any $c \in \mathbb{C}$, the two categories $\mathbf{SV_1}(\mathbf{c}, *)$ and $\mathbf{SG}^+(\mathbf{c}, *)$ are isomorphic (and hence $\mathbf{SV_2}(\mathbf{c}, *)$ and $\mathbf{SG}^+(\mathbf{c}, *)$ are isomorphic).*

Sketch of proof: We first define a functor $F_{SG^+} : \mathbf{SG}^+(\mathbf{c}, *) \to \mathbf{SV_1}(\mathbf{c}, *)$. Given a supergeometric vertex operator superalgebra (V, ν) with rank c, define the vacuum $\mathbf{1}_\nu \in \bar{V}$ by

$$\mathbf{1}_\nu = \nu_0(\mathbf{0});$$

an element $\tau_\nu \in \bar{V}$ by

$$\tau_\nu = \frac{\partial}{\partial \epsilon} \nu_0(\mathbf{0}, \{0, -\epsilon, 0, 0, 0...\});$$

the vertex operator $Y_\nu(v_1, (x, \varphi)) = \sum_{n \in \mathbb{Z}} (v_1)_n x^{-n-1} + \varphi \sum_{n \in \mathbb{Z}} (v_1)_{n-\frac{1}{2}} x^{-n-1}$ associated with $v_1 \in V$ by

$$(v_1)_n v_2 + \theta(v_1)_{n-\frac{1}{2}} v_2 = \operatorname{Res}_z \left(z^n \nu_2((z, \theta); \mathbf{0}, (1, \mathbf{0}), (1, \mathbf{0}))(v' \otimes v_1 \otimes v_2) \right),$$

where Res_z means taking the residue at the singularity $z = 0$, i.e., taking the coefficient of z^{-1}.

It is easy to see from the sewing and grading axioms that $\mathbf{1}_\nu, \tau_\nu \in V$, and in fact, $(V, Y_\nu(\cdot, (x, \varphi)), \mathbf{1}_\nu, \tau_\nu)$ is a vertex operator superalgebra with rank c. The functor F_{SG^+} is defined by

$$F_{SG^+}(V, \nu) = (V, Y_\nu(\cdot, (x, \varphi)), \mathbf{1}_\nu, \tau_\nu) \quad \text{and} \quad F_{SG^+}(\gamma) = \gamma.$$

We next define a functor $F_{SV}^+ : \mathbf{SV_1}(\mathbf{c}, *) \to \mathbf{SG}^+(\mathbf{c}, *)$. Given a vertex operator superalgebra $(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau)$ with rank c and positive square root structure, we want to define maps $(\nu_n^+)^Y : SK^+(n) \to S\mathcal{F}_V(n)$, $Q \mapsto (\nu_n^+)^Y(Q)$. For a supersphere with three tubes $Q = ((z, \theta); (A^{(0)}, M^{(0)}), (a_0^{(1)}, A^{(1)}, M^{(1)}), (a_0^{(2)}, A^{(2)}, M^{(2)}))$, we define $(\nu_2^+)^Y(Q)$ by

$$(\nu_2^+)^Y(Q)(v' \otimes v_1 \otimes v_2) = \langle e^{-\sum_{j \in \mathbb{Z}_+} (A_j^{(0)} L'(j) + M_{j-\frac{1}{2}}^{(0)} G'(j-\frac{1}{2}))} v',$$

$$Y(e^{-\sum_{j \in \mathbb{Z}_+} (A_j^{(1)} L(j) + M_{j-\frac{1}{2}}^{(1)} G(j-\frac{1}{2}))} \cdot (a_0^{(1)})^{-L(0)} \cdot v_1, (x_1, \varphi_1)) \cdot$$

$$\left. e^{-\sum_{j \in \mathbb{Z}_+} (A_j^{(2)} L(j) + M_{j-\frac{1}{2}}^{(2)} G(j-\frac{1}{2}))} \cdot (a_0^{(2)})^{-L(0)} \cdot v_2 \rangle \right|_{(x_i, \varphi_i) = (z_i, \theta_i)}.$$

For elements $Q \in SK^+(n)$, $n \neq 2$, we define $(\nu_n^+)^Y(Q)$ similarly using correlation functions of products of vertex operators, appropriately interpreting the expressions as supermeromorphic superfunctions. The pair $(V, (\nu^+)^Y)$ is a supergeometric vertex operator superalgebra with positive square root structure. The proof of this fact uses Propositions 3.2 and 3.3. The functor F_{SV}^+ is defined by

$$F_{SV}^+(V, Y(\cdot, (x, \varphi)), \mathbf{1}, \tau) = (V, (\nu^+)^Y) \quad \text{and} \quad F_{SV}^+(\gamma) = \gamma.$$

In fact, $F_{SG^+} \circ F_{SV}^+ = 1_{\mathbf{SV_1(c,*)}}$ and $F_{SV}^+ \circ F_{SG^+} = 1_{\mathbf{SG^+(c,*)}}$.　■

By Proposition 5.2 and Theorem 7.1, we can define the isomorphisms of categories $F_{SG^-} = J^- \circ F_{SG^+} : \mathbf{SG}^-(\mathbf{c},*) \to \mathbf{SV_1}(\mathbf{c},*)$ and $F_{SV}^- = F_{SG^-}^{-1}$. In particular,

Corollary 7.2 *For any $c \in \mathbb{C}$, the two categories $\mathbf{SV_1}(\mathbf{c},*)$ and $\mathbf{SG}^-(\mathbf{c},*)$ are isomorphic (and hence $\mathbf{SV_2}(\mathbf{c},*)$ and $\mathbf{SG}^-(\mathbf{c},*)$ are isomorphic). Moreover, the spin structure symmetry between $\mathbf{SG}^+(\mathbf{c},*)$ and $\mathbf{SG}^-(\mathbf{c},*)$ defines an automorphism J_1 of the category $\mathbf{SV_1}(\mathbf{c},*)$ and an automorphism J_2 of the category $\mathbf{SV_2}(\mathbf{c},*)$ given by*

$$J_1(V, Y(\cdot, (x,\varphi)), \mathbf{1}, \tau) = (V, Y^-(\cdot, (x,\varphi)), \mathbf{1}, -\tau) \quad and \quad J_1(\gamma) = \gamma,$$

where $Y^-(v, (x,\varphi)) = Y(v, (x, -\varphi))$, and

$$J_2(V, Y(\cdot, x), \mathbf{1}, \tau) = (V, Y(\cdot, x), \mathbf{1}, -\tau) \quad and \quad J_2(\gamma) = \gamma,$$

respectively. In particular, we have the following commutative diagram of categories and isomorphisms

$$
\begin{array}{ccccc}
\mathbf{SG}^+(\mathbf{c},*) & \overset{F_{SG+},F_{SV}^+}{\longleftrightarrow} & \mathbf{SV_1}(\mathbf{c},*) & \overset{F_1,F_2}{\longleftrightarrow} & \mathbf{SV_2}(\mathbf{c},*) \\
\big\Updownarrow J^\pm & & \big\Updownarrow J_1 & & \big\Updownarrow J_2 \\
\mathbf{SG}^-(\mathbf{c},*) & \overset{F_{SG-},F_{SV}^-}{\longleftrightarrow} & \mathbf{SV_1}(\mathbf{c},*) & \overset{F_1,F_2}{\longleftrightarrow} & \mathbf{SV_2}(\mathbf{c},*) \quad .
\end{array}
$$

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